

Stationary Mean Field Games systems defined on networks

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Abstract

We consider a stationary Mean Field Games system defined on a network. In this framework, the transition conditions at the vertices play a crucial role: the ones here considered are based on the optimal control interpretation of the problem. We prove separately the well-posedness for each of the two equations composing the system. Finally, we prove existence and uniqueness of the solution of the Mean Field Games system.

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1 Introduction

The theory of Mean Field Games (briefly, MFG) has been introduced in [20, 21] to describe the asymptotic behavior of stochastic differential game problems (Nash equilibria) as the number of players tends to $+\infty$. From a mathematical point of view, MFG theory leads to the study of a coupled system of two differential equations: one equation is of Hamilton-Jacobi-Bellman (briefly, HJB) type and it describes the optimal behavior of the single agent, while the other one is a Fokker-Planck (briefly, FP) equation governing the distribution of the overall population. The system can be completed with different boundary conditions (periodic, Dirichlet, Neumann) and initial conditions (initial-terminal condition, planning problem). Existence and uniqueness of strong and weak solutions to the MFG system have been obtained under rather general assumptions on the data of the problems ([9, 8, 15, 26]).

Aim of this note is to study MFG systems defined on a network. While the differential equations are defined in the usual way along the edges, a crucial issue is

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to find the correct conditions at the vertices (*transition conditions*). We will choose a set of transition conditions according to the optimal control interpretation of the system.

Starting with the seminal paper by Lumer [22], a general theory for linear and semilinear differential equations on networks has been developed mainly employing the variational structure of the problem. In this framework the natural transition conditions are, besides the continuity of the solution, the so-called Kirchhoff conditions on the first order derivatives (see [23, 24, 25, 28]).

For a single nonlinear equation, existence and uniqueness results are available only for some specific classes of operators such as conservation law [11] and some Hamilton-Jacobi equations [7]. Therefore the first step in our analysis is to establish existence and uniqueness results for each of the two equations composing the MFG system on the network. Because of its control theoretic interpretation (see [13, 14] and Section 2) the natural transition condition for the HJB equation is the Kirchhoff condition, while for the FP equation it is natural to require the conservation of the flux at the vertices. In Section 2 we discuss the relationship between the transition conditions for the two equations.

After having solved the two equations separately, we tackle our second and main issue: the study of the MFG system on a network. We shall obtain existence of a solution by a fixed point argument; moreover we shall get uniqueness of the solution adapting a classical argument to this framework and taking advantage of the relation between the transition conditions of the two equations.

As far as we know, this is the first paper to consider general MFG systems (HJB equation and FP equation as well) on networks. Indeed, in [6] only a particular class of MFG systems on networks was addressed; in that setting, by a suitable change of variables, the HJB and the FP equation are transformed in two heat equations both with Kirchhoff transition conditions coupled via the initial data. Moreover, it is worth to observe that the papers [18], [16] and [17] consider MFG systems on graphs (namely, the state variable belongs to a discrete set).

We remark that the results here contained can be generalized in several directions (nonlocal coupling, evolutive problems, boundary conditions, weak solutions, ramified spaces etc); moreover the assumptions are far from being optimal. Since this is a first approach to the study of MFG systems on networks, we have tried to keep the presentation as simple as possible in order to avoid technical complications and to concentrate on the network aspects of the problems.

The paper is organized as follows. In the rest of this section, we shall introduce the definition of networks. In Section 2 we give a formal derivation of MFG systems on networks and, especially, of the transition conditions. Section 3 is devoted to our main results for the HJB equation, the FP equation and, mainly, the MFG system. Finally, in the Appendix A we collect some technical results.

Networks. The network $\Gamma = (\mathcal{V}, \mathcal{E})$ is a finite collection of points $\mathcal{V} := \{v_i\}_{i \in I}$ in \mathbb{R}^n connected by continuous, non self-intersecting edges $\mathcal{E} := \{e_j\}_{j \in J}$. Each edge $e_j \in \mathcal{E}$ is parametrized by a smooth function $\pi_j : [0, l_j] \rightarrow \mathbb{R}^n$, $l_j > 0$. Given $v_i \in \mathcal{V}$,

we denote by $Inc_i := \{j \in J : v_i \in e_j\}$ the set of edges branching out from v_i and by $d_{v_i} := |Inc_i|$ the degree of v_i . A vertex v_i is said a boundary vertex if $d_{v_i} = 1$, otherwise it is said a transition vertex. For simplicity, in this paper we will assume that the set of boundary vertices is empty.

For a function $u : \Gamma \rightarrow \mathbb{R}$ we denote by $u_j : [0, l_j] \rightarrow \mathbb{R}$ the restriction of u to e_j , i.e. $u(x) = u_j(y)$ for $x \in e_j$, $y = \pi_j^{-1}(x)$, and by $\partial_j u(v_i)$ the oriented derivative of u at v_i along the arc e_j defined by

$$\partial_j u(v_i) = \begin{cases} \lim_{h \rightarrow 0^+} (u_j(h) - u_j(0))/h, & \text{if } v_i = \pi_j(0); \\ \lim_{h \rightarrow 0^+} (u_j(l_j - h) - u_j(l_j))/h, & \text{if } v_i = \pi_j(l_j). \end{cases}$$

The integral of a function u on Γ is defined by

$$\int_{\Gamma} u(x) dx := \sum_{j \in J} \int_0^{l_j} u_j(r) dr.$$

The space $L^p(\Gamma)$, $p \geq 1$, is the set of functions $u : \Gamma \rightarrow \mathbb{R}$ such that $u_j \in L^p(0, l_j)$ for all $j \in J$ and $\|u\|_p := \sum_{j \in J} \|u_j\|_{L^p(0, l_j)} < \infty$. For $p \geq 1$ and for an integer $m > 0$, we define the Sobolev space $W^{m,p}(\Gamma)$ as the space of continuous functions on Γ such that $u_j \in W^{m,p}(0, l_j)$ for all $j \in J$ and $\|u\|_{m,p} := \sum_{j \in J} \|u_j\|_{W^{m,p}(0, l_j)} < \infty$. As usual we set $H^k(\Gamma) := W^{k,2}(\Gamma)$, $k \in \mathbb{N}$. The space $C^k(\Gamma)$, $k \in \mathbb{N}$, consists of all the continuous functions $u : \Gamma \rightarrow \mathbb{R}$ such that $u_j \in C^k([0, l_j])$ for $j \in J$ and $\|u\|_{C^k} = \max_{\beta \leq k} \|\partial^\beta u\|_{L^\infty} < \infty$. Observe that no continuity condition at the vertices is prescribed for the derivatives neither for a function $u \in W^{m,p}(\Gamma)$ nor for a function $u \in C^k(\Gamma)$.

Finally the space $C^{k,\alpha}(\Gamma)$, for $k \in \mathbb{N}$ and $\alpha \in (0, 1)$, is the space of functions $u \in C^k(\Gamma)$ such that $\partial^k u_j \in C^{0,\alpha}([0, l_j])$ for any $j \in J$ with the norm

$$\|u\|_{C^{k,\alpha}} := \|u\|_{C^k} + \sup_{j \in J} \sup_{x, y \in [0, l_j]} [|\partial^k u_j(x) - \partial^k u_j(y)| / |x - y|^\alpha].$$

2 A formal derivation of the MFG system

The MFG system can be deduced from two different points of view (see [21]): either as the characterization of a Pareto equilibrium for dynamic games with a large number of (indistinguishable) players; or as the optimality conditions for an optimal control problem whose dynamic is governed by a PDE. We explain the two different points of view for MFG systems on networks showing that they lead to the same transition conditions.

Pareto equilibrium: Consider a population of indistinguishable agents, distributed at time $t = 0$ according to the probability m_0 ; any agent moves on a network Γ and its dynamics inside the edge e_j is governed by the stochastic differential equation

$$dX_s = -\gamma_s ds + \sqrt{2\nu_j} dW_s,$$

where γ is the control, $\nu_j > 0$ and W_t is a 1-dimensional Brownian motion. When the agent reaches a vertex $v_i \in \mathcal{V}$, it almost surely spends zero time at v_i and enters in one of the incident edges, say e_j with $j \in \text{Inc}_i$, with probability β_{ij} where

$$\beta_{ij} > 0, \quad \sum_{j \in \text{Inc}_i} \beta_{ij} = 1.$$

(see [13, 14] for a rigorous definition of stochastic processes on networks). The cost criterion is given by

$$\mathbb{E}_x \left[\int_0^T \{L(X_t, \gamma_t) + V[m(X_t)]\} dt + V_0[m(X_T)] \right]$$

where m represents the distribution of the overall population of players. A formal application of the dynamic programming principle gives that the value function u of the previous control problem satisfies

$$\begin{cases} -u_t - \nu_j \partial^2 u + H_j(x, \partial u) = V[m], & (x, t) \in e_j \times (0, T), j \in J \\ \sum_{j \in \text{Inc}_i} \alpha_{ij} \nu_j \partial_j u(v_i, t) = 0 & (v_i, t) \in \mathcal{V} \times (0, T), \\ u_j(v_i, t) = u_k(v_i, t), & j, k \in \text{Inc}_i, (v_i, t) \in \mathcal{V} \times (0, T), \\ u(x, T) = V_0[m(T)], & x \in \Gamma \end{cases} \quad (2.1)$$

where $\alpha_{ij} := \beta_{ij} \nu_j^{-1}$ and the Hamiltonian is given on the edge e_j by

$$H_j(x, p) = \sup_{\gamma} [-\gamma \cdot p - L_j(x, \gamma)].$$

Note that the differential equation inside e_j is defined in terms of the coordinate parametrizing the edge. The second equation in (2.1) is known as the Kirchhoff transition condition and it is consequence of the assumption on the behavior of X_t at the vertices (see [14]). The third line amounts to the continuity at transition vertices.

In order to derive the equation satisfied by the distribution m of the agents, we follow a duality argument. Consider the linearized Hamilton-Jacobi equation

$$\begin{cases} -w_t - \nu \partial^2 w + \partial_p H(x, \partial u) \partial w = 0, & (x, t) \in e_j \times (0, T), j \in J \\ \sum_{j \in \text{Inc}_i} \nu_j \alpha_{ij} \partial_j w(v_i, t) = 0 & (v_i, t) \in \mathcal{V} \times (0, T) \\ w_j(v_i, t) = w_k(v_i, t), & j, k \in \text{Inc}_i, (v_i, t) \in \mathcal{V} \times (0, T), \\ w(x, T) = 0 & x \in \Gamma \end{cases} \quad (2.2)$$

Writing the weak formulation of (2.2) for a test function m , integrating by parts along each edge and regrouping the boundary terms corresponding to the same vertex v_i , we get

$$\begin{aligned} 0 &= \sum_{j \in J} \int_0^T \int_{e_j} (-w_t - \nu_j \partial^2 w + \partial_p H_j(x, \partial u) \partial w) m \, dx dt \\ &= \sum_{j \in J} \left(\int_{e_j} [wm]_0^T dx + \int_0^T \int_{e_j} [m_t - \nu_j \partial^2 m - \partial(m \partial_p H_j(x, \partial u))] w \, dx dt \right) \\ &\quad - \int_0^T \sum_{v_i \in \mathcal{V}} \left[\sum_{j \in \text{Inc}_i} \nu_j m_j(v_i, t) \partial_j w(v_i, t) - (\nu_j \partial_j m(v_i, t) + \partial_p H(v_i, \partial u) m_j(v_i, t)) w(v_i, t) \right] dt. \end{aligned}$$

By the previous identity we obtain that m satisfies inside the edges the equation

$$m_t - \nu \partial^2 m - \partial(m \partial_p H(x, \partial u)) = 0.$$

Moreover, recalling the transition condition for w , the first one of the three terms computed at the transition vertices vanishes if

$$\frac{m_j(v_i, t)}{\alpha_{ij}} = \frac{m_k(v_i, t)}{\alpha_{ik}}, \quad j, k \in Inc_i, (v_i, t) \in \mathcal{V} \times (0, T). \quad (2.3)$$

The vanishing of the other two terms for each $v_i \in \mathcal{V}$, namely

$$\sum_{j \in Inc_i} \nu_j \partial_j m(v_i, t) + \partial_p H(v_i, \partial u) m_j(v_i, t) = 0, \quad (2.4)$$

gives the transition condition for m at the vertices $v_i \in \mathcal{V}$. Note that (2.4) gives the conservation of the total flux of the density m at the vertex v_i (see [11] for a similar condition). Summarizing, for $\nu := \{\nu_j\}_{j \in J}$, we get the system

$$\left\{ \begin{array}{ll} -u_t - \nu \partial^2 u + H(x, Du) = V[m] & (x, t) \in \Gamma \times (0, T) \\ m_t - \nu \partial^2 m - \partial(m \partial_p H(x, \partial u)) = 0 & (x, t) \in \Gamma \times (0, T) \\ \sum_{j \in Inc_i} \alpha_{ij} \nu_j \partial_j u(v_i, t) = 0 & (v_i, t) \in \mathcal{V} \times (0, T) \\ \sum_{j \in Inc_i} \nu_j \partial_j m(v_i, t) + \partial_p H(v_i, \partial u) m_j(v_i, t) = 0 & (v_i, t) \in \mathcal{V} \times (0, T) \\ u_j(v_i, t) = u_k(v_i, t), \frac{m_j(v_i, t)}{\alpha_{ij}} = \frac{m_k(v_i, t)}{\alpha_{ik}} & j, k \in Inc_i, (v_i, t) \in \mathcal{V} \times (0, T) \\ u(x, T) = V_0[m(T)], \quad m(x, 0) = m_0(x) & x \in \Gamma \end{array} \right. \quad (2.5)$$

with the normalization condition $\int_{\Gamma} m(x) dx = 1$. The transition conditions (continuity and either Kirchhoff condition or conservation of total flux) for u and m give d_{v_i} linear conditions for each functions at a $v_i \in \mathcal{V}$, hence they univocally determine the values $u_j(v_i, t)$ and $m_j(v_i, t)$, $j \in Inc_i$.

Optimal control: We consider the *planning problem* for a MFG system, i.e. we prescribe the initial and the terminal condition for the distribution m (see [1, 21]). Consider the functional

$$\inf_{b, m} \int_0^T \int_{\Gamma} \{L(x, b) m + W[m]\} dx dt \quad (2.6)$$

subject to

$$\left\{ \begin{array}{ll} m_t - \nu \partial^2 m - \partial(bm) = 0 & (x, t) \in \Gamma \times (0, T) \\ \sum_{j \in Inc_i} \nu_j \partial_j m(v_i, t) + b(v_i, t) m_j(v_i, t) = 0 & (v_i, t) \in \mathcal{V} \times (0, T) \\ \frac{m_j(v_i, t)}{\alpha_{ij}} = \frac{m_k(v_i, t)}{\alpha_{ik}} & (v_i, t) \in \mathcal{V} \times (0, T) j, k \in Inc_i, \\ m(x, 0) = m_0(x), \quad m(x, T) = m_T(x), & x \in \Gamma. \end{array} \right. \quad (2.7)$$

The problem of minimizing (2.6) under the constraints (2.7) is equivalent to

$$\inf_{b,m} \sup_u \int_0^T \int_{\Gamma} \{L(x,b)m + W[m] - u(m_t - \nu \partial^2 m - \partial(mb))\} dx dt \quad (2.8)$$

where u is the multiplier. We argue as in [1, section 3.3] (see also [15, section 2.5.1]); integrating by part in (2.8), taking into account the transition conditions in (2.7) and minimizing with respect to b , we obtain a minimum problem whose optimality conditions give, at a formal level, a system similar to (2.5) with an initial-terminal condition for m with $V = W'$.

Similar considerations in both the approaches can be used for deriving the stationary (*ergodic*) MFG system

$$\left\{ \begin{array}{ll} -\nu \partial^2 u + H(x, \partial u) + \rho = V[m] & x \in \Gamma \\ \nu \partial^2 m + \partial(m \partial_p H(x, \partial u)) = 0 & x \in \Gamma \\ \sum_{j \in Inc_i} \alpha_{ij} \nu_j \partial_j u(v_i) = 0 & v_i \in \mathcal{V} \\ \sum_{j \in Inc_i} [\nu_j \partial_j m(v_i) + \partial_p H_j(v_i, \partial_j u) m_j(v_i)] = 0 & v_i \in \mathcal{V} \\ u_j(v_i) = u_k(v_i), \frac{m_j(v_i)}{\alpha_{ij}} = \frac{m_k(v_i)}{\alpha_{ik}} & j, k \in Inc_i, v_i \in \mathcal{V} \\ \int_{\Gamma} u(x) dx = 0, \quad \int_{\Gamma} m(x) dx = 1 & \end{array} \right. \quad (2.9)$$

where $\rho \in \mathbb{R}$ is also an unknown.

In the rest of the paper we will only consider the stationary system (2.9). Moreover we will restrict to the case in which all the coefficients in the transition condition for u are equal, i.e.

$$\alpha_{ij} = \alpha_{ik} \quad \forall i \in I, j, k \in Inc_i. \quad (2.10)$$

If (2.10) is not satisfied, the function m should be discontinuous at v_i and this fact clearly involves additional difficulties. In fact it is well known that, in its standard definition, the domain of the Laplace operator on a network is given by the $H^2(\Gamma)$ functions (in particular, continuous) satisfying Kirchhoff condition at the vertices ([14, 22]). Moreover, the continuity condition at transition vertices seems to be a crucial ingredient for the comparison principle (see [29, 24]).

3 Main results

This section contains our main results on the solvability of HJB equations, FP equations and, above all, MFG systems on networks. To this end, we first introduce some assumptions. Consider an Hamiltonian $H : \Gamma \times \mathbb{R} \rightarrow \mathbb{R}$, namely a collection

of operators $(H_j)_{j \in J}$ with $H_j : [0, l_j] \times \mathbb{R} \rightarrow \mathbb{R}$. For some δ and C positive numbers, we assume

$$H_j \in C^2([0, l_j] \times \mathbb{R}); \quad (3.1)$$

$$H_j(x, \cdot) \text{ is convex in } p \text{ for each } x \in [0, l_j]; \quad (3.2)$$

$$\delta|p|^2 - C \leq H_j(x, p) \leq C|p|^2 + C \text{ for } (x, p) \in [0, l_j] \times \mathbb{R}; \quad (3.3)$$

$$\nu = \{\nu_j\}_{j \in J}, \quad \nu_j \in \mathbb{R} \quad \text{with } 0 < \nu_0 := \inf_{j \in J} \nu_j. \quad (3.4)$$

These assumptions will hold throughout this paper unless it is explicitly assumed in a different way. Let us note that no continuity condition for H is required at the vertices and that, clearly, also the diffusion ν may present discontinuities at these points.

Let us now state our result for MFG systems, whose proof is contained in Section 3.3; in Sections 3.1 and 3.2 we shall establish our result for HJB equations and respectively for FP equations.

Theorem 3.1 *Assume (3.1)-(3.4) and that V is a local C^1 coupling, namely*

$$V[m](x) = V(m(x)) \text{ with } V \in C^1([0, +\infty)). \quad (3.5)$$

Then, there exists a solution $(u, m, \rho) \in C^2(\Gamma) \times C^2(\Gamma) \times \mathbb{R}$ to

$$\left\{ \begin{array}{ll} -\nu \partial^2 u + H(x, \partial u) + \rho = V[m] & x \in \Gamma \\ \nu \partial^2 m + \partial(m \partial_p H(x, \partial u)) = 0 & x \in \Gamma \\ \sum_{j \in Inc_i} \nu_j \partial_j u(v_i) = 0 & v_i \in \mathcal{V} \\ \sum_{j \in Inc_i} [\nu_j \partial_j m(v_i) + \partial_p H_j(v_i, \partial_j u) m_j(v_i)] = 0 & v_i \in \mathcal{V} \\ u_j(v_i) = u_k(v_i), m_j(v_i) = m_k(v_i) & j, k \in Inc_i, v_i \in \mathcal{V} \\ \int_{\Gamma} u(x) dx = 0, \quad \int_{\Gamma} m(x) dx = 1, \quad m \geq 0. & \end{array} \right. \quad (3.6)$$

Moreover if

$$\int_{\Gamma} (V[m_1] - V[m_2])(m_1 - m_2) dx \leq 0 \Rightarrow m_1 = m_2, \quad (3.7)$$

then the solution is unique.

Remark 3.1 *As already pointed out in the introduction, this result can be easily adapted to the case of networks having a boundary by imposing Neumann or Dirichlet boundary condition.*

3.1 On the Hamilton-Jacobi-Bellman equation

This section is devoted to the ergodic HJB problem: find $(u, \rho) \in C^2(\Gamma) \times \mathbb{R}$ such that

$$\begin{cases} -\nu \partial^2 u + H(x, \partial u) + \rho = f(x), & x \in \Gamma \\ u_j(v_i) = u_k(v_i), \quad \sum_{j \in \text{Inc}_i} \nu_j \partial_j u(v_i) = 0 & j, k \in \text{Inc}_i, \quad v_i \in \mathcal{V} \end{cases} \quad (3.8)$$

with the normalization condition

$$\int_{\Gamma} u(x) dx = 0. \quad (3.9)$$

Theorem 3.2 *Assume (3.1)-(3.4) and $f \in C^{0,\alpha}(\Gamma)$ for some $\alpha \in (0, 1)$. Then, there exists a unique couple $(u, \rho) \in C^2(\Gamma) \times \mathbb{R}$ satisfying (3.8)-(3.9). Moreover $u \in C^{2,\alpha}(\Gamma)$ and there holds*

$$\|u\|_{C^{2,\alpha}(\Gamma)} \leq C, \quad |\rho| \leq \max_{\Gamma} |H(\cdot, 0) - f(\cdot)| \quad (3.10)$$

with C depending only on $\|f\|_{C^{0,\alpha}}$ and the constants in (3.3)-(3.4).

PROOF Proposition A.1 ensures that, for any $\lambda \in (0, 1)$, there exists a solution $u_{\lambda} \in C^{2,\alpha}(\Gamma)$ to

$$\begin{cases} -\nu \partial^2 u + H(x, \partial u) + \lambda u = f(x), & x \in \Gamma \\ u_j(v_i) = u_k(v_i), \quad \sum_{j \in \text{Inc}_i} \nu_j \partial_j u(v_i) = 0 & j, k \in \text{Inc}_i, \quad v_i \in \mathcal{V}. \end{cases} \quad (3.11)$$

We want to pass to the limit for $\lambda \rightarrow 0$ in (3.11). We first observe that if C_0 is a constant such that $\max_{\Gamma} |H(\cdot, 0) - f(\cdot)| \leq C_0$, then the functions \underline{u} , \bar{u} defined by $\underline{u}(x) = -C_0/\lambda$, $\bar{u}(x) = C_0/\lambda$ for any $x \in \Gamma$, are respectively a sub- and a supersolution of (3.11). By Proposition A.2 we get

$$-C_0 \leq \lambda u_{\lambda}(x) \leq C_0 \quad \text{for any } x \in \Gamma. \quad (3.12)$$

Now, let us introduce the function $w_{\lambda} := u_{\lambda} - \min_{\Gamma} u_{\lambda}$; it is a $C^{2,\alpha}$ solution to

$$-\nu \partial^2 w_{\lambda} + H(x, \partial w_{\lambda}) + \lambda u_{\lambda} = f(x) \quad x \in \Gamma \quad (3.13)$$

with the same continuity and Kirchhoff transition conditions as in (3.11). We claim

$$\|\partial w_{\lambda}\|_{L^2(\Gamma)} \leq C_1 \quad (3.14)$$

for some constant C_1 depending only on $\|f\|_{C^{0,\alpha}}$ and the constants in (3.3)-(3.4) (in particular, independent of λ). Indeed, integrating equation (3.13) on Γ (i.e. using $\phi = 1$ as test function for w_{λ}), we get

$$\int_{\Gamma} H(x, \partial w_{\lambda}) dx + \int_{\Gamma} (\lambda u_{\lambda}) dx = \int_{\Gamma} f dx.$$

By assumption (3.3) and estimate (3.12), we infer

$$\delta \int_{\Gamma} |\partial w_{\lambda}|^2 dx \leq \int_{\Gamma} (C - \lambda u_{\lambda} + f) dx \leq (C + C_0 + \|f\|_{\infty})|\Gamma|$$

which amounts to our claim (3.14). We claim now that

$$\|w_{\lambda}\|_{C^{2,\alpha}(\Gamma)} \leq C_2 \quad (3.15)$$

for some constant C_2 with the same feature of C_1 . To this end, we note that, since w_{λ} is a classical solution to (3.13), by (3.3)-(3.4) and (3.12), there holds

$$\nu_0 |\partial^2 w_{\lambda}| \leq |H(x, \partial w_{\lambda})| + |\lambda u_{\lambda}| \leq C(|\partial w_{\lambda}|^2 + 1) + C_0$$

and, by (3.14)

$$\|\partial^2 w_{\lambda}\|_{L^1(\Gamma)} \leq C_3 \quad (3.16)$$

for some constant C_3 sharing the same features of C_1 . Taking into account (3.14) and (3.16), (possibly increasing C_3) we infer $\|\partial w_{\lambda}\|_{C^{0,\alpha}(\Gamma)} \leq C_3$; using again (3.13) we accomplish the proof of our claim (3.15).

Possibly passing to a subsequence, we may assume that, as $\lambda \rightarrow 0^+$, the sequence $\{w_{\lambda}\}_{\lambda}$ converges to some function $u \in C^{2,\alpha}(\Gamma)$ (observe that u still verifies the continuity and the Kirchhoff conditions) and that $\{\lambda \min_{\Gamma} u_{\lambda}\}_{\lambda}$ converges to some constant ρ . Passing to the limit in (3.13), we get that the couple (u, ρ) satisfies (3.8). Possibly adding a constant to u , we also get (3.9).

To show the uniqueness of ρ , assume that there exist two solutions (u_i, ρ_i) , $i = 1, 2$, of (3.8) and let x_0 be a maximum point of $u_1 - u_2$. If $x_0 \in e_j$, we have $\partial_j u_1(x_0) = \partial_j u_2(x_0)$ and $\partial_j^2 u_1(x_0) \leq \partial_j^2 u_2(x_0)$. Hence using the equation we conclude that $\rho_2 \leq \rho_1$. If $x_0 = v_i$, there holds $\partial_j u_1(x_0) \leq \partial_j u_2(x_0)$ for any $j \in Inc_i$. In fact, we have: $\partial_j u_1(x_0) = \partial_j u_2(x_0)$ for any $j \in Inc_i$; indeed, assuming by contradiction $\partial_k u_1(x_0) < \partial_k u_2(x_0)$ for some $k \in Inc_i$, we get $\sum_{j \in Inc_i} \nu_j \partial_j u_1(x_0) < \sum_{j \in Inc_i} \nu_j \partial_j u_2(x_0)$ which contradicts the Kirchhoff condition. Hence

$$\nu_j \partial_j^2 (u_2 - u_1)(x_0) \geq H(x_0, \partial_j u_2(x_0)) - H(x_0, \partial_j u_1(x_0)) + \rho_2 - \rho_1 = \rho_2 - \rho_1$$

which, together with $\partial_j u_1(x_0) = \partial_j u_2(x_0)$, gives a contradiction for $\rho_2 > \rho_1$. Hence $\rho_2 \leq \rho_1$ and by symmetry $\rho_2 = \rho_1$.

Having proved the uniqueness of ρ , the uniqueness of a solution to (3.8)-(3.9) can be proved as in [7, Corollary 3.1]. Finally, since there exists a unique solution to (3.8) we conclude that all the sequence $(w_{\lambda}, \lambda u_{\lambda})$ converges to (u, ρ) . \square

3.2 On the Fokker-Planck equation

This section is devoted to the following problem

$$\begin{cases} \nu \partial^2 m + \partial(b(x) m) = f(x) & x \in \Gamma \\ m_j(v_i) = m_k(v_i), \quad \sum_{j \in Inc_i} [b(v_i) m_j(v_i) + \nu_j \partial_j m(v_i)] = 0 & j, k \in Inc_i, v_i \in \mathcal{V} \end{cases} \quad (3.17)$$

with the normalization conditions

$$m \geq 0, \quad \int_{\Gamma} m(x) dx = 1. \quad (3.18)$$

Definition 3.1 (i) A strong solution of (3.17) is a function $m \in C^2(\Gamma)$ which satisfies (3.17) in pointwise sense.

(ii) A weak solution of (3.17) is a function $m \in H^1(\Gamma)$ such that

$$\sum_{j \in J} \int_{e_j} (\nu_j \partial_j m + b(x)m) \partial_j \phi dx + \int_{\Gamma} f \phi dx = 0 \quad \forall \phi \in H^1(\Gamma). \quad (3.19)$$

Remark 3.2 By standard arguments, one can easily check that if $m \in C^2(\Gamma)$ is a weak solution to (3.17), then it is also a strong solution to (3.17).

Theorem 3.3 Assume $b \in C^1(\Gamma)$. Then, there exist a unique weak solution m to (3.17)-(3.18) with $f = 0$ and it verifies

$$\|m\|_{H^1} \leq C, \quad 0 < m(x) \leq C \quad (3.20)$$

where the constant C depends only on $\|b\|_{\infty}$ and ν . Moreover $m \in C^2(\Gamma)$ and it is also a strong solution to (3.17).

PROOF We shall proceed adapting the arguments of [4, Theorem II.4.3]. By Proposition A.3, there exists a unique (up to multiplicative constant) solution to problem (3.17). So we only have to prove that this family of solutions contains a (unique) function m satisfying (3.18) which moreover will verify (3.20).

For $\lambda \in (1, +\infty)$ and for any $\phi \in L^{\infty}(\Gamma)$, we set

$$U_{\lambda}(t, x) := U(\lambda t, x) \quad \text{for } (t, x) \in (0, 1) \times \Gamma$$

where U is the solution of the parabolic Cauchy problem (A.15) with $\psi = \phi$ (see Lemma A.2). Observe that U_{λ} solves

$$\partial_t U_{\lambda} - \lambda \nu \partial^2 U + \lambda b \partial U = 0 \quad \text{in } (0, 1) \times \Gamma \quad (3.21)$$

with the same transition conditions and initial datum of (A.15). We claim that

$$U_{\lambda} \text{ are uniformly bounded in } L^2(0, 1; H^1(\Gamma)) \cap L^{\infty}((0, 1) \times \Gamma). \quad (3.22)$$

Indeed, since $\pm \|\phi\|_{\infty}$ are respectively a super- and a subsolution to (3.21), we get

$$|U_{\lambda}(t, x)| \leq \|\phi\|_{\infty} \quad \text{a.e. in } (0, 1) \times \Gamma. \quad (3.23)$$

In other words, we get that the functions U_{λ} are uniformly bounded in $L^{\infty}((0, 1) \times \Gamma)$ and, in particular, in $L^2((0, 1) \times \Gamma)$. On the other hand, using U_{λ} as test function

for problem (3.21), we get

$$\begin{aligned}
\int_{\Gamma} U_{\lambda}^2(\tau, x) dx &+ \lambda \sum_j \nu_j \iint_{(0,1) \times e_j} (\partial_j U_{\lambda})^2 dx dt \\
&= \int_{\Gamma} \phi^2(x) dx - \lambda \iint_{(0,1) \times \Gamma} b \partial U_{\lambda} U_{\lambda} dx dt \\
&\leq \int_{\Gamma} \phi^2(x) dx + \lambda \|b\|_{\infty} \iint_{(0,1) \times \Gamma} |\partial U_{\lambda}| |U_{\lambda}| dx dt.
\end{aligned}$$

Applying Cauchy inequality to the last term (recall that Γ has finite measure), by (3.23), we get

$$\lambda \sum_j \nu_j \iint_{(0,1) \times e_j} (\partial_j U_{\lambda})^2 dx dt \leq c(\lambda + 1)$$

for some constant c independent of λ . We infer that ∂U_{λ} are uniformly bounded in $L^2((0, 1) \times \Gamma)$; thus our claim (3.22) is completely proved.

The property (3.22) yields that there exists a subsequence of $\{U_{\lambda}\}$ (that we still denote by U_{λ}) such that

$$U_{\lambda} \rightarrow \xi \quad \text{in } L^{\infty}((0, 1) \times \Gamma) \text{ weak-}^* \text{ and in } L^2(0, 1; H^1(\Gamma)) \text{ weak as } \lambda \rightarrow +\infty.$$

Let us now use $\beta\theta$ as test function for (3.21), with $\beta \in C_0^{\infty}((0, 1))$ and $\theta \in C^{\infty}(\Gamma)$ (recall: this means that $\theta \in C^0(\Gamma)$ and $\theta \in C^{\infty}(e_j)$ for every $j \in J$); we obtain

$$\frac{1}{\lambda} \iint_{[0,1] \times \Gamma} U_{\lambda} \beta' \theta dx dt + \int_{[0,1]} \beta \sum_j \int_{e_j} (\nu_j \partial_j U_{\lambda} \partial_j \theta + b \partial_j U_{\lambda} \theta) dx dt = 0.$$

Passing to the limit as $\lambda \rightarrow +\infty$, we get

$$\int_{[0,1]} \beta \sum_j \int_{e_j} (\nu_j \partial_j \xi \partial_j \theta + b \partial_j \xi \theta) dx dt = 0.$$

By the arbitrariness of β we get that, for a.e. $t \in (0, 1)$, there holds

$$\sum_j \int_{e_j} (\nu_j \partial_j \xi \partial_j \theta + b \partial_j \xi \theta) dx = 0;$$

namely, ξ is a weak solution to

$$\begin{cases} -\nu \partial^2 \xi + b(x) \partial \xi = 0 & x \in \Gamma \\ \xi_j(v_i) = \xi_k(v_i), \quad \sum_{j \in \text{Inc}_i} \nu_j \partial_j \xi(v_i) = 0 & j, k \in \text{Inc}_i, v_i \in \mathcal{V}. \end{cases}$$

The maximum principle for this problem (see [24, Theorem 2.1]) ensures that the function ξ is independent of x , namely $\xi = \xi(t)$.

On the other hand, using the function m introduced in Proposition A.3 as test function for (3.21), we infer

$$\int_{\Gamma} U_{\lambda}(\tau, x)m(x) dx = \int_{\Gamma} \phi(x)m(x) dx;$$

as $\lambda \rightarrow +\infty$, we get

$$\xi(t) \int_{\Gamma} m(x) dx = \int_{\Gamma} \phi(x)m(x) dx.$$

By the arbitrariness of ϕ (recall also that m cannot be identically zero because it belongs to a 1-dimensional family), we deduce that $\int m dx$ cannot be zero. Moreover, we also infer that ξ is independent of t , namely ξ is a constant. By Proposition A.3, we can choose m such that $\int m dx = 1$. In conclusion, the last equality reads as

$$\xi = \int_{\Gamma} \phi(x)m(x) dx$$

for every $\phi \in L^{\infty}(\Gamma)$ (clearly, ξ depends on ϕ). By Lemma A.2, a standard application of the Kantorovich-Vulikh theorem (see, [23, Theorem 6.3] and the subsequent discussion) ensures that the semigroup associated to the Cauchy problem (A.15) has a strictly positive integral kernel. Therefore, we may accomplish the proof following the same arguments of [4] and of [3, Lemma 2.3].

Finally, let us prove that m belongs to $C^2(\Gamma)$; fix $j \in J$ and rewrite the equation as

$$\nu \partial_j^2 m = -m \partial_j b - b \partial_j m \quad (3.24)$$

Since $b \in C^1(\Gamma)$ and $m \in H^1(\Gamma)$, it follows that $m \in H^2(\Gamma)$, hence $\partial_j m$ is continuous. Therefore by (3.24) we conclude that $m \in C^2(\Gamma)$ and it is also a strong solution to (3.17). \square

3.3 Proof of Theorem 3.1

PROOF OF THEOREM 3.1 Consider the set $\mathcal{K} = \{\mu \in C^{0,\alpha}(\Gamma) : \int_{\Gamma} \mu dx = 1\}$ and observe that \mathcal{K} is a closed subset of the Banach space $C^{0,\alpha}(\Gamma)$. We define an operator $T : \mathcal{K} \rightarrow \mathcal{K}$ according to the scheme

$$\mu \rightarrow u \rightarrow m \quad (3.25)$$

as follows. Given $\mu \in \mathcal{K}$, we solve the problem (3.8)-(3.9) with $f(x) = V[\mu](x)$ for the unknowns u and ρ , which are uniquely defined by Theorem 3.2. Then, for u given, we seek a function m which solves problem (3.17)-(3.18) with $b(x) = \partial_p H(x, \partial u)$. By Theorem 3.3, the function m is univocally defined and we set $m = T(\mu)$. We claim that

$$\text{the map } T \text{ is continuous with compact image.} \quad (3.26)$$

Let $\mu_n, \mu \in \mathcal{K}$ be such that $\|\mu_n - \mu\|_{C^{0,\alpha}} \rightarrow 0$ for $n \rightarrow \infty$ and let $(u_n, \rho_n), (u, \rho)$ be the solutions of (3.8)-(3.9) corresponding to $f(\cdot) = V(\mu_n(\cdot))$ and, respectively,

$f(\cdot) = V(\mu(\cdot))$. By the estimate (3.10), (possibly passing to a subsequence) u_n converges in $C^2(\Gamma)$ to a function \bar{u} and ρ_n converges to some constant $\bar{\rho}$. Since $V[\mu_n] \rightarrow V[\mu]$ in $C^{0,\alpha}$ and since the transition and the boundary conditions pass to the limit by the C^2 -convergence we get that \bar{u} is a solution of (3.8)-(3.9) with $f(x)$ and ρ replaced respectively by $V[m](x)$ and $\bar{\rho}$. By the uniqueness of the solution to (3.8)-(3.9) we get that $\bar{u} = u$ and $\rho = \bar{\rho}$; moreover, all the sequence $\{(u_n, \rho_n)\}$ converges to (u, ρ) . Let m_n and m be respectively the solutions of (3.17)-(3.18) with $f = 0$ for $b = \partial_p H(x, \partial u_n)$ and for $b = \partial_p H(x, \partial u)$. By the estimate (3.20), the functions m_n are equibounded in $H^1(\Gamma)$. Since $\partial_p H(x, \partial u_n)$ uniformly converges to $\partial_p H(x, \partial u)$, passing to the limit in the weak formulation yields that (possibly passing to a subsequence) m_n converges to a solution \bar{m} to (3.17)-(3.18) with $b(x) = \partial_p H(x, \partial u)$. Theorem 3.3 entails: $m = \bar{m}$; hence, the whole sequence $\{m_n\}$ converges to m .

To prove the compactness of the image of T , consider a sequence μ_n such that $\|\mu_n\|_{C^{0,\alpha}} \leq 1$ and let u_n and m_n the functions obtained according to the scheme (3.25). By (3.10), $\|u_n\|_{C^{2,\alpha}}$ is uniformly bounded and therefore by (3.20) also $\|m_n\|_{C^{0,\alpha}}$ is uniformly bounded. As in the proof of Proposition 3.3, we get an uniform bound on the H^2 -norm of m_n for any n . By the compact immersion of $H^2(\Gamma)$ in $H^1(\Gamma)$, we get that the sequence m_n is compact in H^1 and therefore in $C^{0,\alpha}$. Hence, claim (3.26) is completely proved.

We can therefore conclude by the Leray-Schauder fixed point Theorem that the map T admits a fixed point, i.e. a solution of system (3.6). Moreover this solution is also smooth by the regularity results in Theorems 3.2 and 3.3.

Finally the uniqueness of the solution to (3.6) under the assumption (3.7) follows by a standard argument in MFG theory adapted to the networks (see [21]). We assume that there exists two solutions (u_1, m_1, ρ_1) and (u_2, m_2, ρ_2) of (3.6). We set $\bar{u} = u_1 - u_2$, $\bar{m} = m_1 - m_2$, $\bar{\rho} = \rho_1 - \rho_2$ and we write the equations for \bar{u} , \bar{m}

$$\begin{cases} -\nu \partial^2 \bar{u} + H(x, \partial u_1) - H(x, \partial u_2) + \bar{\rho} - (V[m_1] - V[m_2]) = 0 \\ \nu \partial^2 \bar{m} + \partial(m_1 \partial_p H(x, \partial u_1) - m_2 \partial_p H(x, \partial u_2)) = 0 \\ \sum_{j \in Inc_i} \nu_j \partial_j \bar{u}(v_i) = 0 \\ \sum_{j \in Inc_i} \nu_j \partial_j \bar{m}(v_i) + (m_1 \partial_p H(v_i, \partial u_1) - m_2 \partial_p H(v_i, \partial u_2)) = 0 \\ \int_{\Gamma} \bar{m} dx = 0, \quad \int_{\Gamma} \bar{u} dx = 0 \end{cases}$$

Multiplying the equation for \bar{m} by \bar{u} and integrating over e_j , we get

$$\begin{aligned} & \int_{e_j} [-\nu_j \partial_j \bar{u} \partial_j \bar{m} - (m_1 \partial_p H_j(x, \partial u_1) - m_2 \partial_p H_j(x, \partial u_2)) \partial_j \bar{u}(x)] dx \\ & + \left[\bar{u}_j (\nu_j \partial_j \bar{m} + m_1 \partial_p H_j(x, \partial u_1) - m_2 \partial_p H_j(x, \partial u_2)) \right]_0^{l_j} = 0. \end{aligned} \quad (3.27)$$

Multiplying the equation for \bar{u} by \bar{m} and integrating over e_j , we get

$$\begin{aligned} & \int_{e_j} \nu_j \partial_j \bar{u} \partial_j \bar{m} + [H_j(x, \partial_j u_1) - H_j(x, \partial_j u_2) + \bar{\rho} - (V[m_1] - V[m_2])] \bar{m}_j dx \\ & + \left[\nu_j \bar{m} \partial_j \bar{u} \right]_0^{l_j} = 0 \end{aligned} \quad (3.28)$$

Adding (3.27) to (3.28), summing over $j \in J$, regrouping the terms corresponding to a same vertex v_i and taking into account the transition and the normalization conditions for \bar{u} and \bar{m} we get

$$\begin{aligned} & \sum_{j \in J} \int_{e_j} (m_1 - m_2)(V[m_1] - V[m_2])dx + \\ & \sum_{j \in J} \int_{e_j} m_1 [H_j(x, \partial_j u_2) - H_j(x, \partial_j u_1) - \partial_p H_j(x, \partial_j u_2) \partial_j (u_2 - u_1)] dx + \\ & \sum_{j \in J} \int_{e_j} m_2 [H_j(x, \partial_j u_1) - H_j(x, \partial_j u_2) - \partial_p H_j(x, \partial_j u_1) \partial_j (u_1 - u_2)] dx = 0. \end{aligned}$$

Since each of the three terms in the previous identity is non-negative, it follows that it must vanish. By (3.7) we get $m_1 = m_2$. By the uniqueness of the solution to (3.8) we finally get $u_1 = u_2$ and $\rho_1 = \rho_2$. \square

A Appendix

A.1 Auxiliary results for HJB equation

In this section, we study the following semilinear problem

$$\begin{cases} -\nu \partial^2 u + H(x, u, \partial u) + \lambda u = 0, & x \in \Gamma \\ u_j(v_i) = u_k(v_i), \quad \sum_{j \in \text{Inc}_i} \nu_j \partial_j u(v_i) = 0 & j, k \in \text{Inc}_i, v_i \in \mathcal{V}. \end{cases} \quad (\text{A.1})$$

As far as we know, this problem has not been tackled before; we shall establish existence, regularity and uniqueness (via comparison principle).

Definition A.1

(i) A strong solution of (A.1) is a function $u \in C^2(\Gamma)$ which satisfies (A.1) in pointwise sense.

(ii) A weak solution of (A.1) is a function $u \in H^1(\Gamma)$ such that

$$\sum_{j \in J} \int_{e_j} (\nu_j \partial_j u \partial_j \phi + H(x, u, \partial_j u) \phi + \lambda u \phi) dx = 0 \quad \text{for any } \phi \in H^1(\Gamma). \quad (\text{A.2})$$

Remark A.1 One can easily check that, if $u \in C^2(\Gamma)$ is a weak solution of (A.1), then it is also a strong solution.

Let us now state our existence result

Proposition A.1 *Assume*

$$\begin{aligned} H_j(\cdot, r, p) & \text{ is measurable in } x, \text{ for any } (r, p) \in \mathbb{R} \times \mathbb{R} \text{ and} \\ H_j(x, \cdot, \cdot) & \text{ is continuous in } (r, p), \text{ for a.e. } x \in (0, l_j) \end{aligned} \quad (\text{A.3})$$

$$|H(x, r, p)| \leq C_0 + b(|r|)|p|^2 \quad \text{for a.e. } (x, r, p) \in \Gamma \times \mathbb{R} \times \mathbb{R} \quad (\text{A.4})$$

$$H(x, r, p) \text{ is not decreasing in } r \text{ for a.e. } (x, r, p) \in \Gamma \times \mathbb{R} \times \mathbb{R} \quad (\text{A.5})$$

$$\lambda > 0, \nu_j \in \mathbb{R} \quad \text{with } 0 < \nu_0 := \inf_{j \in J} \nu_j \quad (\text{A.6})$$

where $C_0 > 0$, $b : \mathbb{R} \rightarrow \mathbb{R}$ is an increasing function. Then there exists a weak solution to (A.1). Moreover

$$\|u\|_{H^1} \leq C$$

with C depending on C_0 , λ and ν_0 .

Moreover, if H belongs to $C^{0,\alpha}(\Gamma \times \mathbb{R} \times \mathbb{R})$ for some $\alpha \in (0, 1)$, then solution u belongs to $C^{2,\alpha}(\Gamma)$ with

$$\|u\|_{C^{2,\alpha}} \leq C_1(1 + \|u\|_{H^1}) \quad (\text{A.7})$$

where C_1 is a constant depending only on C_0 , b and ν_0 .

Lemma A.1 *Assume (A.3), (A.5) and, for some $C_H > 0$,*

$$|H(x, r, p)| \leq C_H \quad \text{for } (x, r, p) \in \Gamma \times \mathbb{R} \times \mathbb{R}.$$

Then there exists a weak solution u to (A.1). Moreover

$$\|u\|_{\infty} \leq \frac{C_H}{\lambda}. \quad (\text{A.8})$$

PROOF Define a map $T : H^1(\Gamma) \rightarrow H^1(\Gamma)$ by taking the weak solution $u = T(v)$ of

$$\begin{cases} -\nu \partial^2 u + \lambda u = -H(x, v, \partial v) & x \in \Gamma \\ u_j(v_i) = u_k(v_i), \quad \sum_{j \in \text{Inc}_i} \nu_j \partial_j u(v_i) = 0 & j, k \in \text{Inc}_i, v_i \in \mathcal{V}. \end{cases} \quad (\text{A.9})$$

(note that existence of a weak solution to (A.9) follows by the theory of sesqui-linear forms, see for instance [24]). Standard estimates implies that T is continuous with compact image, hence by the Schauder's Theorem it admits a fixed point which is a weak solution to (A.1).

Even though the proof of estimate (A.8) is standard, for completeness we sketch the argument. Let $G : \mathbb{R} \rightarrow \mathbb{R}$ be a smooth function such that $G(t) = 0$ for $t \in (-\infty, 0]$ and G strictly increasing for $t \in (0, \infty)$. Set $K = C_H/\lambda$ and $\phi = G(u - K)$. Then $\phi \in H^1(\Gamma)$ and by taking ϕ as test function in (A.2) we get

$$0 = \sum_{j \in J} \int_{e_j} [\nu_j (\partial_j u)^2 G'(u - K) + (H_j(x, u, \partial_j u) + \lambda K) G(u - K) + \lambda (u - K) G(u - K)] dx$$

Since $H_j(x, u, \partial u) + \lambda K \geq 0$ and $G(u - K) \geq 0$ a.e. on Γ , then

$$\sum_{j \in J} \int_{e_j} \lambda(u - K)G(u - K)dx \leq 0.$$

and by $tG(t) \geq 0$ for $t \in \mathbb{R}$, it follows that $(u - K)G(u - K) = 0$ a.e. on Γ , hence $u \leq K$ a.e. in Γ . \square

PROOF OF PROPOSITION A.1 The proof of the existence result follows exactly the same argument of the proof of [5, Theorem 2.1] replacing their steps 1 and 2 with Lemma A.1. In fact, in the weak formulation of (3.11), given in Definition A.1, the transition condition is transparent and the estimates necessary to prove the result are obtained using the weak formulation (A.2) which is the same of the problem posed in an Euclidean domain.

Consider now $H \in C^{0,\alpha}(\Gamma \times \mathbb{R} \times \mathbb{R})$. We already know that $u \in C^0(\Gamma)$ and we have only to show that $u_j \in C^{2,\alpha}(0, l_j)$ for any $j \in J$ (recall that u_j is the restriction of u to the edge e_j). For j fixed, the equation for u (in distributional sense) is

$$-\nu_j \partial_j^2 u = -\lambda u_j - H(x, u_j, \partial_j u) \quad \text{in } (0, l_j). \quad (\text{A.10})$$

Since $u_j \in C^0([0, l_j])$ and, by (A.4), $H(\cdot, u_j(\cdot), \partial_j u(\cdot)) \in L^1((0, l_j))$, by (A.10) we immediately get $u_j \in W^{2,1}((0, l_j))$ and therefore $\partial_j u \in L^p((0, l_j))$, for any $p \geq 1$. We deduce $H(\cdot, u_j(\cdot), \partial_j u(\cdot)) \in L^p((0, l_j))$ and in particular $u_j \in W^{2,p}((0, l_j))$. Hence $\partial_j u \in C^{0,\alpha}((0, l_j))$ and again by (A.10) we get the statement. Moreover, the estimate (A.7) easily follows from (A.10). \square

Proposition A.2 Assume (A.3) and (A.5)-(A.6). Let the functions $u_1, u_2 \in C^2(\Gamma)$ satisfy

$$\begin{cases} -\nu \partial^2 u_1 + H(x, u_1, \partial u_1) + \lambda u_1 \geq -\nu \partial^2 u_2 + H(x, u_2, \partial u_2) + \lambda u_2 & x \in \Gamma \\ \sum_{j \in \text{Inc}_i} \nu_j \partial_j u_1(v_i) \leq \sum_{j \in \text{Inc}_i} \nu_j \partial_j u_2(v_i) & v_i \in \mathcal{V}. \end{cases} \quad (\text{A.11})$$

Then, $u_1 \geq u_2$ on Γ .

PROOF We argue by contradiction assuming $\max_{\Gamma}(u_2 - u_1) =: \delta > 0$. Let x_0 be a point where $u_2 - u_1$ attains its maximum. The point x_0 either belongs to some edge or it is a vertex. Assume that x_0 belongs to some edge e_j . By their regularity, the functions u_1 and u_2 fulfill

$$u_2(x_0) = u_1(x_0) + \delta, \quad \partial_j u_2(x_0) = \partial_j u_1(x_0), \quad \partial_j^2 u_2(x_0) \leq \partial_j^2 u_1(x_0).$$

In particular, we deduce

$$\begin{aligned} & -\nu_j \partial_j^2 u_1(x_0) + H(x_0, u_1(x_0), \partial_j u_1(x_0)) + \lambda u_1(x_0) \\ & \leq -\nu_j \partial_j^2 u_2(x_0) + H(x_0, u_2(x_0), \partial_j u_2(x_0)) + \lambda(u_2(x_0) - \delta) \\ & < -\nu_j \partial_j^2 u_2(x_0) + H(x_0, u_2(x_0), \partial_j u_2(x_0)) + \lambda u_2(x_0) \end{aligned}$$

which contradicts the first relation in (A.11). Assume that $x_0 = v_i$ for some $v_i \in \mathcal{V}$. Being regular, the functions u_1 and u_2 fulfill $\partial_j u_2(v_i) \leq \partial_j u_1(v_i)$. We claim $\partial_j u_2(v_i) = \partial_j u_1(v_i)$ for each $j \in \text{Inc}_i$. In order to prove this equality we proceed by contradiction and we assume that $\partial_j u_2(v_i) < \partial_j u_1(v_i)$ for some $j \in \text{Inc}_i$. In this case we get $\sum_{j \in \text{Inc}_i} \nu_j \partial_j u_2(v_i) < \sum_{j \in \text{Inc}_i} \nu_j \partial_j u_1(v_i)$ which contradicts the second inequality in (A.11); therefore, our claim is proved. Moreover, since $u_1(v_i) = u_2(v_i) - \delta$, we deduce

$$\begin{aligned} H(v_i, u_1(v_i), \partial_j u_1(v_i)) + \lambda u_1(v_i) &= H(v_i, u_2(v_i) - \delta, \partial_j u_2(v_i)) + \lambda(u_2(v_i) - \delta) \\ &< H(v_i, u_2(v_i), \partial_j u_2(v_i)) + \lambda u_2(v_i). \end{aligned}$$

Taking into account the regularity of H and of u_i ($i = 1, 2$), we infer that in a sufficiently small neighborhood $B_\eta(v_i)$ there holds

$$H(x, u_1(x), \partial u_1(x)) + \lambda u_1(x) < H(x, u_2(x), \partial u_2(x)) + \lambda u_2(x).$$

This inequality and the first relation in (A.11) entail

$$\begin{aligned} \nu_j \partial_j^2 (u_2 - u_1)(v_i) &\geq H(v_i, u_2(v_i), \partial_j u_2(v_i)) - H(x, u_1(v_i), \partial_j u_1(v_i)) \\ &\quad + \lambda(u_2(v_i) - u_1(v_i)) > 0 \end{aligned}$$

which, together with $\partial_j u_2(v_i) = \partial_j u_1(v_i)$, contradicts that $u_2 - u_1$ attains a maximum in $x_0 = v_i$. \square

A.2 Auxiliary results for the Fokker-Planck equation

Proposition A.3 *Assume that $f \in C^0(\Gamma)$ and $b \in C^1(\Gamma)$. The problem (3.17) with $f = 0$ admits a unique (up to multiplicative constant) weak solution. Moreover, for $f \neq 0$, the problem has a solution provided that $\int_\Gamma f dx = 0$.*

PROOF We shall proceed following the technique of [24, Theorem 2.2] and of [4, Theorem II.4.2]. To this end, it is expedient to introduce the following forms on $H^1(\Gamma)$:

$$a(u, v) := \sum_{j \in J} \int_{e_j} (\nu_j \partial_j u \partial_j v + u b_j \partial_j v) dx, \quad (u, v) := \sum_{j \in J} \int_{e_j} u_j v_j dx.$$

We observe that, for $s > 0$ sufficiently large, the form

$$a_s(u, v) := a(u, v) + s(u, v) \tag{A.12}$$

is coercive on $H^1(\Gamma)$. Actually, the regularity of b entails

$$\begin{aligned} a_s(u, u) &\geq \sum_{j \in J} \int_{e_j} [\nu_j (\partial_j u)^2 + s u_j^2 - \|b\|_\infty |u_j| |\partial_j u|] dx \\ &\geq \sum_{j \in J} \int_{e_j} \left[\frac{\nu_j}{2} (\partial_j u)^2 + \left(s - \frac{\|b\|_\infty^2}{\nu_j} \right) u_j^2 \right] dx. \end{aligned}$$

Fix $s > 0$ such that a_s is coercive on $H^1(\Gamma)$. Invoking Lax-Milgram theorem, we obtain that for every $f \in L^2(\Gamma)$, the problem

$$a_s(u, v) = (f, v) \quad \forall v \in H^1(\Gamma) \quad (\text{A.13})$$

has exactly one solution $u =: G_s(f)$. In other words, there exists a map $G_s : L^2(\Gamma) \rightarrow H^1(\Gamma)$ such that $G_s(f)$ is the unique solution in $H^1(\Gamma)$ of problem (A.13). In fact we claim that

$$G_s(f) \in H^2(\Gamma) \quad \forall f \in L^2(\Gamma).$$

Actually, let us observe that, on each edge e_j , the weak formulation of problem (3.17) is equivalent to the equality (in distributional sense)

$$\nu_j \partial_j^2 m = f + \partial_j(bm)$$

where the right-hand side is in $L^2(e_j)$. Whence, $m \in H^2(e_j)$ for every $j \in J$ and our claim is completely proved. Let us observe that the weak formulation is equivalent to

$$(I - sG_s)(u) = G_s(f) \quad \text{in } L^2(\Gamma);$$

indeed, the weak formulation can be written as: $a_s(u, v) = (f, v) + s(u, v)$; hence $u = G_s(f) + sG_s(u)$. We observe that, by the Rellich-Kondrachov theorem (see [24]), G_s maps compactly $L^2(\Gamma)$ into itself. By Fredholm alternative, the existence and the uniqueness of our problem are related to the properties of the operator $(I - sG_s^*)$ where G_s^* is the adjoint operator of G_s .

Let us now calculate G_s^* . To this end, it is expedient to introduce the problem ($s > 0$ is the same as before)

$$\begin{cases} sw - \nu \partial^2 w + b(x) \partial w = h & x \in \Gamma \\ w_j(v_i) = w_k(v_i), \quad \sum_{j \in \text{Inc}_i} \nu_j \partial_j m(v_i) = 0 & v_i \in \mathcal{V}, j, k \in \text{Inc}_i \end{cases} \quad (\text{A.14})$$

whose weak formulation is

$$\sum_{j \in J} \int_{e_j} (\nu_j \partial_j w \partial_j v + b_j \partial_j w v + s w v) dx = \sum_{j \in J} \int_{e_j} h_j v dx \quad \forall v \in H^1(\Gamma).$$

Arguing as before, we infer that there exists a compact map $\tilde{G}_s : L^2(\Gamma) \rightarrow H^2(\Gamma)$ such that $\tilde{G}_s(h)$ is the unique weak solution to problem (A.14).

We claim that $G_s^* = \tilde{G}_s$. In order to prove this claim, it suffices to show that there holds $(G_s(f), h) = (f, \tilde{G}_s(h))$ for every $f, h \in L^2(\Gamma)$. For $m := G_s(f)$ and $z := \tilde{G}_s(h)$, the regularizing effect of G_s and \tilde{G}_s ensures

$$\begin{aligned} (f, \tilde{G}_s(h)) &= \int_{\Gamma} f z dx = \sum_j \int_{e_j} [sm - \nu_j \partial_j^2 m - \partial(bm)] z dx \\ &= \sum_j \int_{e_j} [smz + \nu_j \partial_j m \partial_j z + bm \partial_j z] dx = \int_{\Gamma} m h dx = \int_{\gamma} G_s(f) h dx \end{aligned}$$

where the third equality is due to the Kirchhoff condition in (3.17), while the last two equalities are due respectively to the weak definition of (A.14) and to the definition of

m . Hence, we accomplished the proof that \tilde{G}_s is the dual of G_s . Therefore, invoking Fredholm alternative, we infer that the dimension of the kernel of $(I - sG_s)$ is finite and it coincides with the one of the kernel of $(I - s\tilde{G}_s)$. In order to evaluate the latter, we observe that it is the space of the weak solution to (A.14) with $h = 0$ and $s = 0$. For this problem, Theorem [24, Theorem 2.1] ensures the maximum principle and in particular that every solution is constant. Therefore the kernel of $(I - sG_s)$ is one-dimensional and the first part of the statement is completely proved. In conclusion, still Fredholm alternative guarantees

$$Rg(I - sG_s) = Ker(I - sG_s^*)^\perp;$$

in particular, problem (3.17) has a solution provided that f is orthogonal to a one-dimensional space. Using $\phi = 1$ as test function in (3.19), we obtain the desired compatibility condition for the solvability of the problem. \square

Lemma A.2 *For every $\phi \in L^2(\Gamma)$, the parabolic Cauchy problem*

$$\left\{ \begin{array}{ll} \partial_t U - \nu \partial^2 U + b \partial U = 0 & \text{in } (0, +\infty) \times \Gamma \\ \sum_{j \in Inc_i} \nu_j \partial_j U(t, v_i) = 0 & v_i \in \mathcal{V}, t \in (0, +\infty) \\ U_j(t, v_i) = U_k(t, v_i) & v_i \in \mathcal{V}, j, k \in Inc_i, t \in (0, +\infty) \\ U(0, x) = \psi(x) & \text{on } \Gamma. \end{array} \right. \quad (\text{A.15})$$

admits exactly one weak solution. Moreover, for $\phi \geq 0$, there holds:

$$U(t, x) > 0 \quad \forall (t, x) \in (0, +\infty) \times \Gamma.$$

PROOF The existence and uniqueness of the solution are established in [29, Theorem 3.4]. For $\phi \geq 0$, the solution U is strictly positive in $(0, +\infty) \times \Gamma$ because the corresponding semigroup is holomorphic, positive and irreducible (see [12] or [2]). We observe that the positivity of the semigroup is a straightforward consequence of the comparison principle. Moreover, the semigroup is holomorphic because the form a_s introduced in (A.12) is coercive (as established in the proof of Proposition A.3). Finally, the irreducibility of the semigroup can be obtained following the same arguments of [19, Proposition 5.2]. \square

Remark A.2 *Let us recall that estimates for the kernel function for problem (A.15) can be obtained arguing as in [27, 10].*

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References

- [1] Y. Achdou, F. Camilli and I. Capuzzo Dolcetta, Mean field games: numerical methods for the planning problem, *SIAM J. Control Optim.* 50 (2012), 77–109.
- [2] W. Arendt, *Heat kernels*, Publication of the 9th Internet Seminar 2005-06 of Ulm University, available at www.uni-ulm.de/fileadmin/website_uni_ulm/mawi.inst.020/arendt/downloads/internetseminar.pdf
- [3] M. Bardi and E. Feleqi, Nonlinear elliptic systems and Mean Field Games, preprint available at cvgmt.sns.it/paper/2655.
- [4] A. Bensoussan, *Perturbation methods in optimal control*, John Wiley and Sons, Chichester 1988.
- [5] L. Boccardo, F. Murat and J.P. Puel, Existence de solutions faibles pour des équations elliptiques quasi-linéaires à croissance quadratique. *Nonlinear partial differential equations and their applications. Collège de France Seminar, Vol. IV (Paris, 1981/1982)*, 19–73, Res. Notes in Math., 84, Pitman, Boston, 1983.
- [6] F. Camilli, E. Carlini and C. Marchi, A model problem for Mean Field Games on networks, *Discrete Contin. Dyn. Syst.* 35 (2015), no. 9, 4173–4192.
- [7] F. Camilli, C. Marchi and D. Schieborn, The vanishing viscosity limit for Hamilton-Jacobi equation on networks. *J. Differential Equations* 254 (2013), 4122–4143.
- [8] P. Cardaliaguet, J. Graber, A. Porretta and D. Tonon, Second order mean field games with degenerate diffusion and local coupling, arXiv:1407.7024, 2014.
- [9] P. Cardaliaguet and J. Graber, Mean field games systems of first order, arXiv:1401.1789, 2014.
- [10] C. Cattaneo, The spread of the potential on a weighted graph, *Rend. Sem. Mat. Univ. Politec. Torino* 57 (1999), no. 4, 221–229.
- [11] G.M. Coclite and M. Garavello, Vanishing viscosity for traffic on networks. *SIAM J. Math. Anal.* 42 (2010), no. 4, 1761–1783.
- [12] K.-J. Engel and R. Nagel, *A short course on operator semigroups*. Springer, New York, 2006.
- [13] M. Freidlin and S. Sheu, Diffusion processes on graphs: stochastic differential equations, large deviation principle. *Probab. Theory Related Fields* 116 (2000), no. 2, 181–220.
- [14] M. Freidlin and A. Wentzell, Diffusion processes on graphs and the averaging principle. *Ann. Probab.* 21 (1993), no. 4, 2215–2245.

- [15] D. Gomes and J. Saude, Mean field games - A brief survey. *Dyn. Games Appl.* 4 (2014), no. 2, 110–154.
- [16] D. Gomes, J. Mohr and R. Souza, Continuous time finite state mean field games. *Appl. Math. Optim.* 68 (2013), no. 1, 99–143.
- [17] D. Gomes, J. Mohr and R. Souza, Discrete time, finite state space mean field games. *J. Math. Pures Appl.* (9) 93 (2010), no. 3, 308–328.
- [18] O. Guéant, Existence and uniqueness result for mean field games with congestion effect on graphs, *Appl. Math. Optim.* (to appear), <http://arxiv.org/abs/1110.3442>.
- [19] M. Kramar Fijavz, D. Mugnolo and E. Sikolya, Variational and semigroup methods for waves and diffusion in networks. *Appl. Math. Optim.* 55 (2007), no. 2, 219–240.
- [20] M. Huang, R.P. Malhamé and P.E. Caines, Large population stochastic dynamic games: closed-loop McKean-Vlasov systems and the Nash certainty equivalence principle. *Commun. Inf. Syst.* 6 (2006), no. 3, 221–251.
- [21] J.-M. Lasry and P.-L. Lions, Mean field games. *Jpn. J. Math.*, 2 (2007), no. 1, 229–260.
- [22] G. Lumer, Espaces ramifiés et diffusions sur les réseaux topologiques. *C. R. Acad. Sci. Paris Sér. A-B* 291 (1980), no. 12, A627–A630.
- [23] D. Mugnolo, *Semigroups methods for evolution equations on networks*, Understanding Complex Systems, Springer-Verlag, Berlin, 2014.
- [24] S. Nicaise, Elliptic operators on elementary ramified spaces. *Integral Equations Operator Theory* 11 (1988), no. 2, 230–257.
- [25] Yu. V. Pokornyi and A. V. Borovskikh, Differential equations on networks (geometric graphs). *J. Math. Sci. (N.Y.)* 119 (2004), no. 6, 691–718.
- [26] A. Porretta, On the Planning Problem for the Mean Field Games System, *Dyn. Games Appl.* (2014), no. 4, 231–256.
- [27] J.P. Roth, Le spectre du laplacien sur un graphe, in *Theorie du potentiel (Orsay, 1983)*, 521–539, Lecture Notes in Math., 1096, Springer, Berlin, 1984.
- [28] J. von Below, Classical solvability of linear parabolic equations on networks. *J. Differential Equations* 72 (1988), no. 2, 316–337.
- [29] J. von Below and S. Nicaise, Dynamical interface transition in ramified media with diffusion, *Comm. Partial Differential Equations* 21 (1996), 255–279.